

A Clifford algebra associated to generalized Fibonacci quaternions

Cristina FLAUT

Abstract. In this paper we find a Clifford algebra associated to generalized Fibonacci quaternions. In this way, we provide a nice algorithm to obtain a division quaternion algebra starting from a quaternion non-division algebra and vice-versa.

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1. Introduction

In 1878, W. K. Clifford discovered Clifford algebras. These algebras generalize the real numbers, complex numbers and quaternions(see [Le; 06]).

The theory of Clifford algebras is intimately connected with the theory of quadratic forms. In the following, we will consider K to be a field of characteristic not two. Let (V, q) be a K -vector space equipped with a nondegenerate quadratic form over the field K . A *Clifford algebra* for (V, q) is a K -algebra C with a linear map $i : V \rightarrow C$ satisfying the property

$$i(x)^2 = q(x) \cdot 1_C, \forall x \in V,$$

such that for any K -algebra A and any K linear map $\gamma : V \rightarrow A$ with $\gamma^2(x) = q(x) \cdot 1_A, \forall x \in V$, there exists a unique K -algebra morphism $\gamma' : C \rightarrow A$ with $\gamma = \gamma' \circ i$.

Such an algebra can be constructed using the tensor algebra associated to a vector space V . Let $T(V) = K \oplus V \oplus (V \otimes V) \oplus \dots$ be the tensor algebra associated to the vector space V and let \mathcal{J} be the two-sided ideal of $T(V)$ generated by all elements of the form $x \otimes x - q(x) \cdot 1$, for all $x \in V$. The associated Clifford algebra is the factor algebra $C(V, q) = T(V) / \mathcal{J}$. ([Kn; 88], [La; 04])

Theorem 1.1. (Poincaré-Birkhoff-Witt). ([Kn; 88], p. 44) *If $\{e_1, e_2, \dots, e_n\}$ is a basis of V , then the set $\{1, e_{j_1}e_{j_2}\dots e_{j_s}, 1 \leq s \leq n, 1 \leq j_1 < j_2 < \dots < j_s \leq n\}$ is a basis in $C(V, q)$.*

The most important Clifford algebras are those defined over real and complex vector spaces equipped with nondegenerate quadratic forms. Every nondegenerate quadratic form over a real vector space is equivalent with the following standard diagonal form:

$$q(x) = x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_s^2,$$

where $n = r + s$ is the dimension of the vector space. The pair of integers (r, s) is called *the signature* of the quadratic form. The real vector space with this quadratic form is usually denoted $\mathbb{R}_{r,s}$ and the Clifford algebra on $\mathbb{R}_{r,s}$ is denoted $Cl_{r,s}(\mathbb{R})$. For other details about Clifford algebras, the reader is referred to [Ki, Ou; 99], [Ko; 10] and [Sm; 91].

Example 1.2.

- i) For $p = q = 0$ we have $Cl_{0,0}(K) \simeq K$;
- ii) For $p = 0, q = 1$, it results that $Cl_{0,1}(K)$ is a two-dimensional algebra generated by a single vector e_1 such that $e_1^2 = -1$ and therefore $Cl_{0,1}(K) \simeq K(e_1)$. For $K = \mathbb{R}$ it follows that $Cl_{0,1}(\mathbb{R}) \simeq \mathbb{C}$.
- iii) For $p = 0, q = 2$, the algebra $Cl_{0,2}(K)$ is a four-dimensional algebra spanned by the set $\{1, e_1, e_2, e_1e_2\}$. Since $e_1^2 = e_2^2 = (e_1e_2)^2 = -1$ and $e_1e_2 = -e_2e_1$, we obtain that this algebra is isomorphic to the division quaternions algebra \mathbb{H} .
- iv) For $p = 1, q = 1$ or $p = 2, q = 0$, we obtain the algebra $Cl_{1,1}(K) \simeq Cl_{2,0}(K)$ which is isomorphic with a split(i.e. nondivision) quaternion algebra, called *paraquaternion algebra* or *anti-quaternion algebra*. (See [Iv, Za; 05])

2. Preliminaries

Let $\mathbb{H}(\beta_1, \beta_2)$ be the generalized real quaternion algebra, the algebra of the elements of the form $a = a_1 \cdot 1 + a_2e_2 + a_3e_3 + a_4e_4$, where $a_i \in \mathbb{R}, i \in \{1, 2, 3, 4\}$, and the elements of the basis $\{1, e_2, e_3, e_4\}$ satisfy the following multiplication table:

\cdot	1	e_2	e_3	e_4
1	1	e_2	e_3	e_4
e_2	e_2	$-\beta_1$	e_4	$-\beta_1e_3$
e_3	e_3	$-e_4$	$-\beta_2$	β_2e_2
e_4	e_4	β_1e_3	$-\beta_2e_2$	$-\beta_1\beta_2$

We denote by $\mathbf{n}(a)$ the norm of a real quaternion a . The norm of a generalized quaternion has the following expression $\mathbf{n}(a) = a_1^2 + \beta_1a_2^2 + \beta_2a_3^2 + \beta_1\beta_2a_4^2$. For $\beta_1 = \beta_2 = 1$, we obtain the real division algebra \mathbb{H} , with the basis $\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji, ik = -ki, jk = -kj$.

Proposition 2.1. ([La; 04], Proposition 1.1) *The quaternion algebra $\mathbb{H}(\beta_1, \beta_2)$ is isomorphic with quaternion algebra $\mathbb{H}(x^2\beta_1, y^2\beta_2)$, where $x, y \in K^*$. \square*

The Fibonacci numbers was introduced by *Leonardo of Pisa (1170-1240)* in his book *Liber abbaci*, book published in 1202 AD (see [Kos; 01], p. 1, 3). This name is attached to the following sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots,$$

with the n th term given by the formula:

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2,$$

where $f_0 = 0, f_1 = 1$.

In [Ho; 61], the author generalized Fibonacci numbers and gave many properties of them:

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where $h_0 = p, h_1 = q$, with p, q being arbitrary integers. In the same paper [Ho; 61], relation (7), the following relation between Fibonacci numbers and generalized Fibonacci numbers was obtained:

$$h_{n+1} = pf_n + qf_{n+1}. \quad (2.1)$$

For the generalized real quaternion algebra, the Fibonacci quaternions and generalized Fibonacci quaternions are defined in the same way:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

for the n th Fibonacci quaternions, and

$$H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4 = pF_n + qF_{n+1}, \quad (2.2)$$

for the n th generalized Fibonacci quaternions.

In the following, we will denote the n th generalized Fibonacci number and a n th generalized Fibonacci quaternion element with $h_n^{p,q}$, respectively $H_n^{p,q}$. In this way, we emphasis the starting integers p and q .

It is known that the expression for the n th term of a Fibonacci element is

$$f_n = \frac{1}{\sqrt{5}}[\alpha^n - \beta^n] = \frac{\alpha^n}{\sqrt{5}}[1 - \frac{\beta^n}{\alpha^n}], \quad (2.3)$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

From the above, we obtain the following limit:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{n}(F_n) &= \lim_{n \rightarrow \infty} (f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1 \beta_2 f_{n+3}^2) = \\ &= \lim_{n \rightarrow \infty} (\frac{\alpha^{2n}}{5} + \beta_1 \frac{\alpha^{2n+2}}{5} + \beta_2 \frac{\alpha^{2n+4}}{5} + \beta_1 \beta_2 \frac{\alpha^{2n+6}}{5}) = \\ &= \text{sgn} E(\beta_1, \beta_2) \cdot \infty, \text{ where } E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)], \\ &\text{since } \alpha^2 = \alpha + 1. (\text{see [Fl, Sh; 13]}) \end{aligned}$$

If $E(\beta_1, \beta_2) > 0$, there exist a number $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $\mathbf{n}(F_n) > 0$. In the same way, if $E(\beta_1, \beta_2) < 0$, there exist a

number $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have $\mathbf{n}(F_n) < 0$. Therefore for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there is a natural number $n_0 = \max\{n_1, n_2\}$ such that $\mathbf{n}(F_n) \neq 0$, hence F_n is an invertible element for all $n \geq n_0$. Using the same arguments, we can compute the following limit:

$$\lim_{n \rightarrow \infty} (\mathbf{n}(H_n^{p,q})) = \lim_{n \rightarrow \infty} (h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1 \beta_2 h_{n+3}^2) = \text{sgn} E'(\beta_1, \beta_2) \cdot \infty, \text{ where } E'(\beta_1, \beta_2) = \frac{1}{5} (p + \alpha q)^2 E(\beta_1, \beta_2), \text{ if } E'(\beta_1, \beta_2) \neq 0. \text{ (see [Fl, Sh; 13])}$$

Therefore, for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there exist a natural number n'_0 such that $\mathbf{n}(H_n^{p,q}) \neq 0$, hence $H_n^{p,q}$ is an invertible element for all $n \geq n'_0$.

Theorem 2.2. ([Fl, Sh; 13], Theorem 2.6) *For all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$, there exist a natural number n' such that for all $n \geq n'$ Fibonacci elements F_n and generalized Fibonacci elements $H_n^{p,q}$ are invertible elements in the algebra $\mathbb{H}(\beta_1, \beta_2)$. \square*

Theorem 2.3. ([Fl, Sh; 13], Theorem 2.1) *The set $\mathcal{H}_n = \{H_n^{p,q} / p, q \in \mathbb{Z}, n \geq m, m \in \mathbb{N}\} \cup \{0\}$ is a \mathbb{Z} -module. \square*

3. Main results

Remark 3.1. We remark that the \mathbb{Z} -module from Theorem 2.3 is a free \mathbb{Z} -module of rank 2. Indeed, $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathcal{H}_n$, $\varphi((p, q)) = H_n^{p,q}$ is a \mathbb{Z} -module isomorphism and $\{\varphi(1, 0) = F_n, \varphi(0, 1) = F_{n+1}\}$ is a basis in \mathcal{H}_n .

Remark 3.2. By extension of scalars, we obtain that $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{H}_n$ is a \mathbb{R} -vector space of dimension two. A basis is $\{\bar{e}_1 = 1 \otimes F_n, \bar{e}_2 = 1 \otimes F_{n+1}\}$. We have that $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{H}_n$ is isomorphic with the \mathbb{R} -vector space $\mathcal{H}_n^{\mathbb{R}} = \{H_n^{p,q} / p, q \in \mathbb{R}\} \cup \{0\}$. A basis in $\mathcal{H}_n^{\mathbb{R}}$ is $\{F_n, F_{n+1}\}$.

Let $T(\mathcal{H}_n^{\mathbb{R}})$ be the tensor algebra associated to the \mathbb{R} -vector space $\mathcal{H}_n^{\mathbb{R}}$ and let $C(\mathcal{H}_n^{\mathbb{R}})$ be the Clifford algebra associated to tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$. From Theorem 1.1, it results that this algebra has dimension four.

Case 1: $\mathbb{H}(\beta_1, \beta_2)$ is a division algebra

Remark 3.3. Since in this case $E(\beta_1, \beta_2) > 0$, for all $n \geq n'$ (as in Theorem 2.2.), then $\mathcal{H}_n^{\mathbb{R}}$ is an Euclidean vector space. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1 F_n + x_2 F_{n+1}$, $w = y_1 F_n + y_2 F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as in the following:

$$\langle z, w \rangle = x_1 y_1 \mathbf{n}(F_n) + x_2 y_2 \mathbf{n}(F_{n+1}).$$

We remark that all properties of inner product are fulfilled. Indeed, since for all $n \geq n'$ we have $\mathbf{n}(F_n) > 0$ and $\mathbf{n}(F_{n+1}) > 0$, it results that $\langle z, z \rangle = x_1^2 \mathbf{n}(F_n) + x_2^2 \mathbf{n}(F_{n+1}) = 0$ if and only if $x_1 = x_2 = 0$, therefore $z = 0$.

On $\mathcal{H}_n^{\mathbb{R}}$ with the basis $\{F_n, F_{n+1}\}$, we define the following quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}} : \mathcal{H}_n^{\mathbb{R}} \rightarrow \mathbb{R}$,

$$q_{\mathcal{H}_n^{\mathbb{R}}}(x_1 F_n + x_2 F_{n+1}) = \mathbf{n}(F_n)x_1^2 + \mathbf{n}(F_{n+1})x_2^2.$$

Let $Q_{\mathcal{H}_n^{\mathbb{R}}}$ be the bilinear form associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$,

$$\begin{aligned} Q_{\mathcal{H}_n^{\mathbb{R}}}(x, y) &= \frac{1}{2}(q_{\mathcal{H}_n^{\mathbb{R}}}(x+y) - q_{\mathcal{H}_n^{\mathbb{R}}}(x) - q_{\mathcal{H}_n^{\mathbb{R}}}(y)) = \\ &= \mathbf{n}(F_n)x_1y_1 + \mathbf{n}(F_{n+1})x_2y_2. \end{aligned}$$

The matrix associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is

$$A = \begin{pmatrix} \mathbf{n}(F_n) & 0 \\ 0 & \mathbf{n}(F_{n+1}) \end{pmatrix}.$$

We remark that $\det A = \mathbf{n}(F_n)\mathbf{n}(F_{n+1}) > 0$, for all $n \geq n'$. Since $E(\beta_1, \beta_2) > 0$, therefore $\mathbf{n}(F_n) > 0$, for $n > n'$. We obtain that the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $Cl_{2,0}(K)$ which is isomorphic to a split quaternion algebra.

From the above results and using Proposition 2.1, we obtain the following theorem:

Theorem 3.4. *If $\mathbb{H}(\beta_1, \beta_2)$ is a division algebra, there is a natural number n' such that for all $n \geq n'$, the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic with the split quaternion algebra $\mathbb{H}(-1, -1)$. \square*

Case 2: $\mathbb{H}(\beta_1, \beta_2)$ is not a division algebra

Remark 3.5. i) If $E(\beta_1, \beta_2) > 0$, then $\mathcal{H}_n^{\mathbb{R}}$ is an Euclidean vector space, for all $n \geq n'$, as in Theorem 2.2. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1 F_n + x_2 F_{n+1}$, $w = y_1 F_n + y_2 F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as in the following:

$$\langle z, w \rangle = x_1 y_1 \mathbf{n}(F_n) + x_2 y_2 \mathbf{n}(F_{n+1}).$$

ii) If $E(\beta_1, \beta_2) < 0$, then $\mathcal{H}_n^{\mathbb{R}}$ is also an Euclidean vector space, for all $n \geq n'$, as in Theorem 2.2. Indeed, let $z, w \in \mathcal{H}_n^{\mathbb{R}}$, $z = x_1 F_n + x_2 F_{n+1}$, $w = y_1 F_n + y_2 F_{n+1}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$. The inner product is defined as in the following:

$$\langle z, w \rangle = -x_1 y_1 \mathbf{n}(F_n) - x_2 y_2 \mathbf{n}(F_{n+1}).$$

We have $\langle z, z \rangle = -x_1^2 \mathbf{n}(F_n) - x_2^2 \mathbf{n}(F_{n+1})$, and since for all $n \geq n'$ we have $\mathbf{n}(F_n) < 0$ and $\mathbf{n}(F_{n+1}) < 0$, it results that $\langle z, z \rangle = -x_1^2 \mathbf{n}(F_n) - x_2^2 \mathbf{n}(F_{n+1}) = 0$ if and only if $x_1 = x_2 = 0$, therefore $z = 0$.

On $\mathcal{H}_n^{\mathbb{R}}$ with the basis $\{F_n, F_{n+1}\}$, we define the following quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}} : \mathcal{H}_n^{\mathbb{R}} \rightarrow \mathbb{R}$,

$$q_{\mathcal{H}_n^{\mathbb{R}}}(x_1 F_n + x_2 F_{n+1}) = q_{\mathcal{H}_n^{\mathbb{R}}}(x_1 F_n + x_2 F_{n+1}) = \mathbf{n}(F_n) x_1^2 + \mathbf{n}(F_{n+1}) x_2^2.$$

Let $Q_{\mathcal{H}_n^{\mathbb{R}}}$ be the bilinear form associated to the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$,

$$\begin{aligned} Q_{\mathcal{H}_n^{\mathbb{R}}}(x, y) &= \frac{1}{2}(q_{\mathcal{H}_n^{\mathbb{R}}}(x+y) - q_{\mathcal{H}_n^{\mathbb{R}}}(x) - q_{\mathcal{H}_n^{\mathbb{R}}}(y)) = \\ &= \mathbf{n}(F_n) x_1 y_1 + \mathbf{n}(F_{n+1}) x_2 y_2. \end{aligned}$$

The matrix associated to quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is

$$A = \begin{pmatrix} \mathbf{n}(F_n) & 0 \\ 0 & \mathbf{n}(F_{n+1}) \end{pmatrix}.$$

We remark that $\det A = \mathbf{n}(F_n) \mathbf{n}(F_{n+1}) > 0$, for all $n \geq n'$.

If $E(\beta_1, \beta_2) > 0$, therefore $\mathbf{n}(F_n) > 0$, for $n > n'$. We obtain that the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $Cl_{2,0}(K)$ which is isomorphic to a split quaternion algebra.

If $E(\beta_1, \beta_2) < 0$, therefore $\mathbf{n}(F_n) < 0$, for $n > n'$. Then the quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite and the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $Cl_{0,2}(K)$ which is isomorphic to the quaternion division algebra \mathbb{H} .

From the above results and using Proposition 2.1, we obtain the following theorem:

Theorem 3.6. *If $\mathbb{H}(\beta_1, \beta_2)$ is not a division algebra, there is a natural number n' such that for all $n \geq n'$, if $E(\beta_1, \beta_2) > 0$, then the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic with the split quaternion algebra $\mathbb{H}(-1, -1)$. If $E(\beta_1, \beta_2) < 0$, then the Clifford algebra associated to the real vector space $\mathcal{H}_n^{\mathbb{R}}$ is isomorphic with the division quaternion algebra $\mathbb{H}(1, 1)$.* \square

Example 3.7. 1) For $\beta_1 = 1, \beta_2 = -1$, we obtain the split quaternion algebra $\mathbb{H}(1, -1)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[-5 - 10\alpha] < 0$ and, for $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 + f_{n+1}^2 - f_{n+2}^2 - f_{n+3}^2 < 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 + f_{n+2}^2 - f_{n+3}^2 - f_{n+4}^2 < 0$, for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $Cl_{0,2}(K)$ which is isomorphic to the quaternion division algebra $\mathbb{H}(1, 1)$.

2) For $\beta_1 = -2, \beta_2 = -3$, we obtain the split quaternion algebra $\mathbb{H}(-2, -3)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[23 + 43\alpha] > 0$. For $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 - f_{n+1}^2 - f_{n+2}^2 + f_{n+3}^2 > 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 - f_{n+2}^2 - f_{n+3}^2 + f_{n+4}^2 > 0$, for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $Cl_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1, -1)$.

3) For $\beta_1 = 2, \beta_2 = -3$, we obtain the split quaternion algebra $\mathbb{H}(2, -3)$. In this case, we have $E(\beta_1, \beta_2) = \frac{1}{5}[-33 - 44\alpha] < 0$. For $n' = 0$, we obtain $\mathbf{n}(F_n) = f_n^2 + 2f_{n+1}^2 - 3f_{n+2}^2 - 6f_{n+3}^2 < 0$, $\mathbf{n}(F_{n+1}) = f_{n+1}^2 + 2f_{n+2}^2 - 3f_{n+3}^2 - 6f_{n+4}^2 > 0$, for all $n \geq 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is negative definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to $Cl_{0,2}(K)$ which is isomorphic to the division quaternion algebra $\mathbb{H}(1, -1)$.

3) For $\beta_1 = \beta_2 = -\frac{1}{2}$, we obtain the split quaternion algebra $\mathbb{H}(-\frac{1}{2}, -\frac{1}{2})$. Therefore $E(\beta_1, \beta_2) = \frac{3}{20} > 0$ and for $n' = 1$ we obtain $\mathbf{n}(F_n) > 0$ and $\mathbf{n}(F_{n+1}) > 0$. The quadratic form $q_{\mathcal{H}_n^{\mathbb{R}}}$ is positive definite, therefore the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $Cl_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1, -1)$.

The algorithm.

1) Let $\mathbb{H}(\beta_1, \beta_2)$ be a quaternion algebra, $\alpha = \frac{1+\sqrt{5}}{2}$ and $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$,

2) Let V be the \mathbb{R} -vector space $\mathcal{H}_n^{\mathbb{R}} = \{H_n^{p,q} / p, q \in \mathbb{R}\} \cup \{0\}$.

3) If $E(\beta_1, \beta_2) > 0$, then the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $Cl_{2,0}(K)$ which is isomorphic to the split quaternion algebra $\mathbb{H}(-1, -1)$.

4) If $E(\beta_1, \beta_2) < 0$, then the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic with $Cl_{0,2}(K)$ which is isomorphic to the division quaternion algebra $\mathbb{H}(1, 1)$.

Conclusions. In this paper, we extend the \mathbb{Z} -module of the generalized Fibonacci quaternions to a real vector space $\mathcal{H}_n^{\mathbb{R}}$. We proved that the Clifford algebra $C(\mathcal{H}_n^{\mathbb{R}})$ associated to the tensor algebra $T(\mathcal{H}_n^{\mathbb{R}})$ is isomorphic to a split quaternion algebra or to a division algebra if $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$ is positive or negative. We also gave an algorithm which allows us to find a division quaternion algebra starting from a split quaternion algebra and vice-versa.

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Cristina FLAUT

Faculty of Mathematics and Computer Science,

Ovidius University,

Bd. Mamaia 124, 900527, CONSTANTA,

ROMANIA

<http://cristinaflaut.wikispaces.com/>

<http://www.univ-ovidius.ro/math/>

e-mail:

cflaut@univ-ovidius.ro

cristina_flaut@yahoo.com